A family of regular quantum interiors for non-rotating black holes I: The GRNSS spacetimes

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Abstract

A seemingly natural mechanism is proposed, that could stop the gravitational collapse of a very massive body. Without needing to change the concept of the collapsing process itself, that is, without invoking thin layers nor resorting to asymptoticity (as has been usually done in the literature), it is proven that a model can be built in which the quantum vacuum is able to produce a negative stress that may stop the collapse of the black hole, reaching a final state of the spacetime structure that is a static de Sitter model. The solution is found by looking into a generic family of spacetimes: that of maximal spherically symmetric ones expanded by a geodesic radial null one-form from flat spacetime. They are called here GRNSS spaces, and are proven to constitute a distinguished family of Kerr-Schild metrics. The models considered previously in the literature are easily recovered in this approach, which yields, moreover, an infinite set of possible candidates for the interior of the black hole. First steps towards their semi-classical quantization are undertaken. It is shown that the quantization protocol may be here more easily carried out than within conformal field theory.

1 Introduction

Soon after the appearance of General relativity, K. Schwarzschild found a solution of Einstein's equations [1] that was useful to describe both the exterior and the interior spacetime metric created by a spherically symmetric body in the case of uniform energy density. This very early model already showed a new feature of General Relativity that still concerns theoretical physicists. According to General Relativity there exists a limit involving the mass and the size of a star above which, in accordance with classical matter properties, it cannot exist and the appearance of a singularity is unavoidable [2]–[5]. In the late 60's, S.W. Hawking and R. Penrose gave a general framework to analyze gravitational collapse, and established their now well-known *Singularity Theorems* (see e.g. [6]). This seemed to settle down the issue, but it has been suggested since then, for some decades, that other effects, most likely those coming from quantum mechanics (see e.g. [7]–[10]) might alter those predictions in a significant way (see e.g. [11]—[17]). Nowadays, the issue of the non-avoidance of singularities of space-time is still far from being resolved. However, there are more and more indications that, indeed, quantum effects may become most principal agents and that a regularized object instead of a true singularity could be formed after the collapse of a very massive object.

As everybody knows, Einstein's equations, or those corresponding to the low energy limits of the usual candidates for suitable quantum theories of gravitation, have two distinguished parts. One corresponds to the geometry and the other to the stress-energy content. It is clear that, for any solution to exist, the perfect matching of both pieces is essential. Due to the difficulty in getting (regularising) the contribution from the stress-energy tensor, it is usually assumed that the spacetime can still be described classically without much loss. In this sense, our work (divided in two papers) gives a general solution for the geometry that may be expected to be born after the collapse of a non-rotating object takes place, following both old and recent ideas on the type of core that is formed and on the precise way this happens (see e.g. [18]–[28]). The case of rotating objects will be considered elsewhere [53].

The case dealt with in the present paper is only the first step of the whole approach, corresponding to the resolution of the left hand side of the aforementioned equations. Its purpose is to begin the process of building up a way to find, within the distinguished (and quite generic) family of solutions that will be obtained in the first part of the paper, some compatible quantum fields that, once regularized, may yield the same result as, at least, a particular energy-momentum tensor inside this family. As we shall see, the variety of

possibilities that comes out, in addition to those of some previous attempts by different authors (also recovered here), and the good compliance with basic quantum requirements, compels us to believe that this goal can be attained and, in particular too, that black hole singularities might in the end be indeed removed by quantum effects.

The present paper deals with the most plausible models, whereas in a second one (referred to as paper II in what follows) it will be proven that they constitute indeed the general solution for the stress-energy that was assumed to exist. It is seen also, that isotropization takes place far from the regularization scale. We will also deal there with the possible extension of the formalism to other candidates, e.g. stringy black holes.

This paper is two-fold. On one side, as mentioned elsewhere, we show in detail how the obtaining of regularized sources for the interiors of non-rotating black holes can be carried out, in fact, without changing the initial concept of the collapsing process itself, i.e. without having to invoke thin layers nor very detailed asymptotic behaviors to get the expected results only there. For the general case of an arbitrary equation of state, we refer the reader to [29], where some explicit solutions are constructed, yet with a different type of energy-momentum tensor than the one we consider here. In Sect. 2 we start with a generic family of spacetimes, which is wide enough to include all those that will be relevant for our purposes. In Sect. 3, we write down the matching conditions in order to couple any pair of spacetimes of the family under study through an arbitrary hypersurface. We get the result that the physical solution can only be a three-sphere of constant "radius". In Sect. 4, we briefly review the impossibility of matching Schwarzschild and de Sitter spacetimes directly. In Sect. 5, we specify the matching conditions for a "classical" model of a Schwarzschild-de Sitter exterior. And in Sect. 6, we write the correct solution for the matching of a de Sitter interior, as the core of the object, with an exterior which fits with current observational data. One easily checks that the de Sitter core is the least convergent regular solution, so that it is the limiting solution between singularity-free and singular cores. In Sect. 7, we prove that the transition expected to take place is accomplished, provided we demand regularity of the collapsed body only. We recover the two known models from our analysis, and show that there is indeed an infinite set of possible interiors (see also [31]). In Sect. 8, we carry out a numerical analysis for all the candidates of the preceding section. The results show that the object is mostly far away from the regularization scale where the notions of space and time lose their sense, and therefore it still may be analyzed with some approximate methods in order to find out possible quantum sources. Before doing this, in Sect. 9, some aspects

regarding the horizons and the topological structure of the solutions are briefly considered. In Sect. 10 we study the energy conditions in these sources, and show that they can be easily related to the properties of the energy density only. In Sect. 11, we start the study of the sources that may yield some of the solutions of the general family. In this respect, we make the first steps towards their (semi-classical) quantization. Preliminary results show that they are indeed good candidates, deserving further analysis, being on the same footing as when one deals with a conformal field theory. At the same time, we show that they fulfill the conditions for an anomaly origin of their sources. We also indicate how quantum field corrections can be immediately incorporated into any initial classical model within the family. A result which tells us, in particular, that quantization may be carried out more easily here than in a conformal quantum field theory. We end up with some final remarks. A brief survey can be found in [32].

2 The family of spacetimes under study

The spacetimes under study are constituted by the family of maximal spherically symmetric spacetimes expanded by a geodesic radial null one-form from flat spacetime (GRNSS spaces). They are defined through the Kerr-Schild metric

$$ds^2 = ds^2_\eta + 2H(r)\boldsymbol{\ell}\otimes\boldsymbol{\ell}, \tag{1}$$

where ds^2_η stands for the flat spacetime metric, H is an arbitrary function of r —the radial coordinate of the spherical symmetry— defined in some open region of the manifold, and $\boldsymbol{\ell}$ is a geodesic radial null one-form. Another expression for the family is (recall that the signature of the metric is $(-1, 1, 1, 1)$)

$$ds^2 = -(1-H)dt^2 + 2H dt\, dr + (1+H)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{2}$$

where the coordinates are the customary spherical ones. In these coordinates, $\boldsymbol{\ell} = (1/\sqrt{2})(dt+dr)$. The other possibility, i.e. $\boldsymbol{\ell} = (1/\sqrt{2})(dt-dr)$ yields the same physical results. Clearly enough, all the metrics have spherical symmetry and, since $H = H(r)$, ∂_t is an integrable Killing vector. In particular, for $H<1$ it is timelike, for $H=1$ null, and for $H>1$ spacelike. The above coordinate choice avoids coordination problems near the possible horizons and Cauchy hypersurfaces, e.g. when $H=1$.

However it is worthwhile to make use of the existence of such integrable Killing vector in order to write down the whole family of metrics in a explicitly static form for the region

where $H < 1$. The final expression is

$$ds^2 = -(1-H)dt_s{}^2 + \frac{1}{1-H}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{3}$$

where dt_s is related to dt by

$$dt_s = dt - \frac{H}{1-H}dr. \tag{4}$$

This last expression for the family looks as a generalization of the Schwarzschild metric and may help to identify the class of spacetimes we are dealing with.[1]

The next task will be to characterize their geometrical properties. To this end we will use an orthonormal cobasis (local observer). In particular, we are going to choose the following, which also avoids any problem at the possible horizons:

$$\begin{aligned} \boldsymbol{\Theta}^0 = \left(1 - \frac{H}{2}\right)dt - \frac{H}{2}dr, \quad \boldsymbol{\Theta}^1 = \left(1 + \frac{H}{2}\right)dr + \frac{H}{2}dt, \\ \boldsymbol{\Theta}^2 = r\, d\theta, \quad \boldsymbol{\Theta}^3 = r \sin\theta\, d\varphi. \end{aligned} \tag{5}$$

The Riemann tensor has the following non-zero components, together with the ones obtained using index symmetry (we follow the sign convention of [4])

$$\begin{aligned} R_{0101} = -\frac{H''}{2}, \quad R_{0202} = R_{0303} = -\frac{H'}{2r}, \quad R_{1212} = R_{1313} = \frac{H'}{2r}, \\ R_{2323} = \frac{H}{r^2}. \end{aligned} \tag{6}$$

The Ricci tensor for these spacetimes has the following non-zero components

$$R_{00} = -R_{11} = -\frac{1}{2}\left(H'' + \frac{2H'}{r}\right), \quad R_{22} = R_{33} = \frac{1}{r}\left(H' + \frac{H}{r}\right). \tag{7}$$

And the scalar curvature is given by

$$R = H'' + \frac{4H'}{r} + \frac{2H}{r^2}. \tag{8}$$

Finally, the Einstein tensor has for components

$$G_{00} = -G_{11} = \frac{1}{r}\left(H' + \frac{H}{r}\right), \quad G_{22} = G_{33} = -\frac{1}{2}\left(H'' + \frac{2H'}{r}\right). \tag{9}$$

Whence, we see that the Einstein tensor satisfies the relations

$$G_{00} + G_{11} = 0, \quad G_{22} = G_{33}. \tag{10}$$

[1] $\boldsymbol{\ell}$ can also be written as $\boldsymbol{\ell} = (1/\sqrt{2})(dt_s + dr^*)$, where $dr^* = dr/(1-H)$ is a direct generalization of the usual (Wheeler) tortoise radial coordinate.

It is now clear that, imposing the additional relation $G_{11} = G_{22}$, the whole set would be invariant under any change of the cobasis, i.e. under any Lorentz transformation (because in that case $G_{\alpha\beta}$ would be proportional to $g_{\alpha\beta}$). In our situation, these relations are invariant under *more restrictive* conditions. Actually under any change of cobasis adapted to the spherical symmetry, i.e. under any Lorentz transformation between $\{\mathbf{\Theta}^0, \mathbf{\Theta}^1\}$ and between $\{\mathbf{\Theta}^2, \mathbf{\Theta}^3\}$. As a consequence, the physical interpretation of all these observers will be exactly the same.[2] This is the reason why they are actually acceptable as consistent conditions to represent an admissible generalization of the absolute isotropic vacuum ($G_{\alpha\beta} \propto g_{\alpha\beta}$)[3] —see e.g. [20, 21]. Of course we could have imposed absolute isotropy in our expressions. The result would then have been $H(r) = Ar^2 + B/r$, where A, B are arbitrary constants. One usually sets $B = 0$ in order to avoid singular behaviors near $r = 0$ (i.e. the Schwarzschild solution), thus recovering the de Sitter spacetime for the isotropic vacuum, as expected. However we have preferred to keep the metric in its more general form. Actually we will show below how a pure de Sitter model is insufficient, for being considered as a reasonable model. Notice, by the way, that the spherical vacuum conditions are here obtained as a direct byproduct of the family of metrics under study and they do not come from any specific imposition.

We shall use Einstein's field equations in the form

$$G_{\alpha\beta} = \frac{8\pi G_N}{c^2} T_{\alpha\beta} - \Lambda g_{\alpha\beta}. \tag{12}$$

Usually, one chooses units for which $G_N = 1$, $c = 1$. If the energy-matter content is of a quantum origin, the r.h.s. should be understood as $< T_{\alpha\beta} >$, and $< \Lambda >$, in accordance with the semiclassical approach to gravitation.

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[2]For instance, in the region with $H < 1$, where the metric can be re-written in the explicit static form

$$ds^2 = -(1-H)dt_s{}^2 + (1-H)^{-1}dr^2 + r^2(d\theta^2 \sin^2\theta \, d\varphi^2), \tag{11}$$

we have the natural, and customarily used, static local observer associated with the cobasis $\mathbf{\Theta}^0 = \sqrt{1-H}dt_s, \mathbf{\Theta}^1 = (1/\sqrt{1-H})dr, \mathbf{\Theta}^2 = rd\theta, \mathbf{\Theta}^3 = r\sin\theta d\varphi$, to interpret the results.

[3]The spacetimes satisfying $G_{00} + G_{11} = 0$, and $G_{22} = G_{33}$ are just of the type being described here or a generalization of the Nariai metric, see e.g. [30]. However the latter cannot be joined with any black hole model unless mass shells are introduced, which is our aim to avoid, since they introduce arbitrariness into the scheme. A detailed account will be given in II.

3 General matching conditions

In this section we will write down the conditions for two metrics of the preceding family to match with each other. This kind of junction conditions constitutes by now a well-established branch of geometry. However, there is no canonical way of imposing them in the literature, and special care has to be taken in order not to confuse the reader. We will follow the formalism contained in [33]–[36].

Our focus will be in matching two metrics with the aim of having one of them as an interior of the other, from some point in time onwards. Here time refers to the time experienced by an observer inside the static region of the spacetime manifold.

The general form of a hypersurface that clearly adjusts itself to the spherical symmetry of any of these spacetimes is as follows

$$\Sigma : \begin{cases} \theta = \lambda_\theta, \\ \varphi = \lambda_\varphi, \\ r = r(\lambda), \\ t = t(\lambda), \end{cases} \tag{13}$$

where $\{\lambda, \lambda_\theta, \lambda_\varphi\}$ are the parameters of the hypersurface. Its tangent vectors are

$$\vec{e}_\theta = \partial_{\lambda_\theta} \overset{\Sigma}{=} \partial_\theta, \quad \vec{e}_\varphi = \partial_{\lambda_\varphi} \overset{\Sigma}{=} \partial_\varphi, \quad \vec{e}_\lambda = \partial_\lambda \overset{\Sigma}{=} \dot{r}\partial_r + \dot{t}\partial_t, \tag{14}$$

where the dot means derivative with respect to λ. The normal one-form is then ($\mathbf{n} \cdot \vec{e}_i = 0$, $i = \{\theta, \varphi, \lambda\}$)

$$\mathbf{n} \overset{\Sigma}{=} \sigma(\dot{r}dt - \dot{t}dr). \tag{15}$$

σ is a free function in the case $\mathbf{n}$ is a null one-form. Otherwise, one can set $\mathbf{n} \cdot \mathbf{n} = \pm 1$, by chosing σ as

$$\sigma_\pm \overset{\Sigma}{=} \frac{\pm 1}{\sqrt{|\dot{t}^2 - \dot{r}^2 + H(\dot{r} - \dot{t})^2|}}. \tag{16}$$

The first junction conditions reduce to the coincidence of the first differential form of Σ at each spacetime. We must thus identify both hypersurfaces in some way. Obviously, the identification of $(\lambda_i)_1$ with $(\lambda_i)_2$ (1 and 2 label each of the spacetimes) is the most natural one, due to the symmetry of the above scheme. This yields

$$\Big[r(\lambda)\Big] = 0, \quad \text{where} \quad [f] \equiv (f_2 - f_1)_\Sigma, \tag{17}$$

$$\Big[\dot{r}^2 - \dot{t}^2 + H(\dot{r} + \dot{t})^2\Big] = 0. \tag{18}$$

The second set of junction conditions comes from (we do not use any mass shell, yet this could be added without problem)

$$\Big[\mathcal{H}_{ij}\Big] = 0, \tag{19}$$

where $\mathcal{H}_{ij}$ is defined by

$$\mathcal{H}_{ij} \overset{\Sigma}{\equiv} -m_\rho\Big(\frac{\partial^2\phi^\rho}{\partial\lambda^i\partial\lambda^j} + \Gamma^\rho_{\mu\nu}\frac{\partial\phi^\mu}{\partial\lambda^i}\frac{\partial\phi^\nu}{\partial\lambda^j}\Big). \tag{20}$$

Here $\vec{m}$ is a vector that completes the set $\{\vec{e}_i\}$ to form a vectorial basis of the manifold.[4] We will choose

$$\vec{m} = \dot{r}\partial_t - \dot{t}\partial_r.$$

This choice has the property that $\vec{m}\cdot\mathbf{n} = \sigma(\dot{r}^2+\dot{t}^2)$. Therefore, if it vanishes, the hypersurface, Σ, becomes degenerate and the joining process itself cannot be carried out. Furthermore, $\phi^\rho(\lambda)$ are the parametric equations of the hypersurface ($\{\phi^0,\phi^1,\phi^2,\phi^3\} = \{t,r,\theta,\varphi\}_\Sigma$), and $\Gamma^\rho_{\mu\nu}$ are the connection coefficients.

Further calculations yield two supplementary conditions, namely

$$[r\dot{t}] = 0, \tag{21}$$

$$\Big[H(\ddot{t}\dot{t} - \ddot{r}\dot{r}) + (1+H)\ddot{r}\dot{t} + (1-H)\ddot{t}\dot{r} - H'\Big(\dot{r}^3 + \frac{\dot{t}^3}{2} + \frac{\dot{r}^2\dot{t}}{2}\Big)\Big] = 0, \tag{22}$$

where H' is the derivative of H with respect to r.

In order to properly close the junction, it is still necessary to choose the signs of σ to be the same in both regions, e.g. to impose a natural interior vs. exterior matching. In order to be sure to deal with all possible candidates, one is forced to look for all different solutions of hypersurfaces and spacetimes in the preceding equations. In fact, it turns out from Eqs. (17)–(22) that the matching conditions translate into[5]

$$[r] = 0, \quad [\dot{t}] = 0, \quad [H] = 0, \quad [H'] = 0.$$

On the other side, H is simply a function of r, so that the two last relations may be viewed as a set of implicit relations defining $r(\lambda)$ in terms of the coefficients of H_1 and H_2. Thus, one arrives at the following conclusion:

[4] For the cases in which $\mathbf{n}$ is non-null, $\vec{m}$ can be chosen simply as $\vec{n}$, and $\mathcal{H}_{ij}$ becomes the second fundamental form. $\mathcal{H}_{ij}$ allows for dealing with a transition at the event horizon of the blak hole.

[5] There is also the possibility of having $t + r =$ const., but this implies $r = 0$ for a finite t, besides representing a geodesic null motion. Therefore they cannot properly descibe the final static and regular interior for the matter of a black hole.

"*The only acceptable hypersurfaces fulfilling the matching conditions, that preserve the spherical symmetry, between two spacetimes of the GNRSS family, are those satisfying* $r_1(\lambda) = r_2(\lambda) = R = const.$, $\dot{t}_1 = \dot{t}_2$, $[H] = [H'] = 0$."

Without losing generality, one can choose $t_1 = t_2 = \lambda$, because of the global existence of the Killing vector ∂_t. Moreover, we realize that the chosen coordinates are privileged ones, in which the matching is explicitly C^1. The hypersurface, Σ, will be timelike, null or spacelike according to $H < 1$, $H = 1$ or $H > 1$, respectively.

Our specific aim in the rest of this work will be to investigate if this situation can be actually met within our family of solutions (GNRSS space) and, also, the physical reliability of the necessary matching condition itself, including the value of H at the matching hypersurface.

Of course, other solutions for the exterior picture of the object, e.g. asymptotically stopping, etc, may be viewed as refined treatments of the whole process. The final results, however, should coincide with the final (stopped) stage we will consider.

4 The effect of the quantum vacuum. Israel's conditions.

Ever since the formulation of the idea that a vacuum state should finish in a mode with an equation of state of the type $\rho + p = 0$, where ρ is the vacuum energy density and p its pressure (or stress), it seemed natural to try to match a Schwarzschild solution, acting as an exterior metric, with a de Sitter solution, acting as the interior one. If we had considered the junction conditions choosing $\vec{l} = \vec{n}$, as is often done, we would have obtained, instead of Eqs. (21) and (22), the following ones, that account for the continuity of the second fundamental form, K_{ij}.[6]

$$\left[\sigma\left\{-\dot{t} + H(\dot{t} + \dot{r})\right\}\right] = 0, \tag{23}$$

$$\left[\sigma\left\{\ddot{r}\dot{t} - \ddot{t}\dot{r} + H'\left(\dot{r}^3 - \frac{\dot{t}^3}{2} + \frac{3}{2}\dot{r}^2\right) + \frac{HH'}{2}[3(\dot{r}^2\dot{t} - \dot{t}^2\dot{r}) + \dot{r}^3 + \dot{t}^3]\right\}\right] = 0. \tag{24}$$

Thus setting $r_1 = r_2 = R$, R a constant, and $H_1 = 2G_N m_1/c^2 r$, $H_2 = (\Lambda_2/3)r^2$ (being m_1, Λ_2 constants with well-known interpretations), we see that the only possibility in order to make them compatible is $H_1 = H_2 = 1$. In principle this yields an interesting relation

[6] Notice that $K_{ij} = \mathcal{H}_{ij}$ taking $\vec{l} = \vec{n}$.

between the exterior mass and the quantum stress

$$\Lambda_2 = 6G_N m_1/c^2R^3. \tag{25}$$

However, if $H_1 = H_2 = 1$, it is clear that the first fundamental form becomes degenerate, i.e. $ds^2_\Sigma = R^2(d\lambda_\theta^2 + \sin\lambda_\theta{}^2 d\lambda_\varphi^2)$. Thus we do not have a hypersurface but actually a *surface*. This is in fact a direct consequence of the presence of a horizon (notice that we have $H = 1$). Its null character makes the above formalism insufficient, and one has to apply the junction conditions on a null hypersurface, or better do it once for all possibilities, as has been the case in the former section. This point went unnoticed in [37], where the authors claimed to have matched Schwarzschild and de Sitter metrics at the horizon. In fact the true result eventually turns out to be negative as well, i.e. Schwarzschild and de Sitter can *never* be directly joined (this is clearly seen from the solution of the former section). This fact impelled other authors to consider a massive layer (usually thin) in order to overcome the problem [18, 19].

A more direct demonstration of this impossibility is through Israel's conditions. Israel's conditions are usually associated with the continuity of the Einstein tensor in a matching situation. Because of Einstein's field equations, they can also be interpreted as conditions on the stress-energy tensor across the hypersurface. They are contained in the preceding ones and have proven to be useful in getting part of the set of restrictions in a very quick way.

In terms of the Einstein tensor, they read

$$\left[n_\rho G^\rho_\alpha\right] = 0. \tag{26}$$

In the exterior region, where we have the Schwarzschild solution, the Einstein tensor is zero, therefore we must have

$$n_\rho (G_2)^\rho_\alpha = 0, \tag{27}$$

being G_2 de Sitter's Einstein tensor which satisfies $G_2 = -\Lambda_2 g$, so that we get

$$\Lambda_2 \mathbf{n} = 0 \tag{28}$$

as a necessary condition. This clearly sets $\Lambda_2 = 0$ and hence a contradiction.

5 The case of a Schwarzschild-de Sitter exterior

We recall that our aim was to find a regular interior of the exterior Schwarzschild geometry that may encompass some of the prevailing quantum ideas. However, it is not wrong to consider as the exterior solution a Schwarzschild–de Sitter model. That is to say, an exterior that may also match with recent observational evidence of a non-zero value for the cosmological constant (see e.g. [38, 39]) as well as uncharged black holes (e.g. [40]).[7]. The exterior metric is then generated by ($G_N = c = 1$)

$$H_1 = \frac{2m_1}{r} + \frac{\Lambda_1}{3} r^2, \tag{29}$$

where m_1 is the usual Schwarzschild mass, and Λ_1 accounts for the cosmological constant. This solution is also known as the Kottler-Trefftz solution, see [41, 42, 30, 43]. As $r \to \infty$ the spacetime tends to a de Sitter model of cosmological constant Λ_1 (see also [20, 21]).

The junction conditions translate then into

$$H_2(R) = H_1(R) = \frac{2m_1}{R} + \frac{\Lambda_1}{3} R^2, \tag{30}$$

$$H_2'(R) = H_1'(R) = -\frac{2m_1}{R^2} + 2\frac{\Lambda_1}{3} R. \tag{31}$$

6 Regular interiors

The preceding conditions are the ones to be imposed in order to have a proper matching of both spacetimes. Moreover, we will also focus on those interior solutions which are everywhere regular. From the expressions of the Riemann tensor and the metric, we see that this may only be accomplished if

$$H_2(0) = 0, \quad H_2'(0) = 0. \tag{32}$$

Thus, we finally encounter four conditions in order to have a regular interior solution. Two of them are imposed on the matching hypersurface, while the other two are imposed at the origin of the spherical symmetry. We shall thus expand H_2 at $r - R$ or r, depending on the situation, or at their corresponding adimensional values $\tilde{r}$ or $\tilde{r} - 1$, where $\tilde{r} = r/R$.

[7] In fact, other more general exterior spacetimes could be chosen within our family of spacetimes. For instance, other type of quantum vacuum contributions, see also Sect. 7.1.4, or other classical solutions, as the Reissner-Nordström one (see Sect. 10). See also the Final Remarks section.

From now on, we will consider H_2 to be an analytic function of the variable $\tilde{r}$, a most natural hypothesis in view of the regular character prescribed for the interior solution.[8] In this case, the origin conditions tell us that

$$H_2(\tilde{r}) = \sum_{n=2}^{\infty} b_n \tilde{r}^n. \tag{33}$$

Now, one has to impose the two other conditions. Obviously it is the same to consider $H_2(\tilde{r})$ or $H_2(\tilde{r}-1)$ in the whole procedure. However, we will first work with $H_2(\tilde{r}-1)$ in order to implement the junction conditions directly. From the preceding result, one immediately has

$$H_2 = \sum_{n=0}^{\infty} a_n (\tilde{r}-1)^n, \tag{34}$$

and the junction conditions tell us that

$$a_0 = H_1(1) = \frac{2m_1}{R} + \frac{\Lambda_1}{3}R^2, \tag{35}$$

$$a_1 = H_1'(1) = -2\left(\frac{m_1}{R} + \frac{\Lambda_1}{3}R^2\right), \tag{36}$$

where $H_1(\tilde{r}) = (2m_1/R)(1/\tilde{r}) + (\Lambda_1 R^2/3)\tilde{r}^2$ and the prime denotes now derivation with respect to $\tilde{r}$.

The following step is to impose regularity of the solution —Eqs. (32). We get

$$\sum_{n=0}^{\infty} (-1)^n a_n = 0 \tag{37}$$

and

$$\sum_{n=0}^{\infty} (-1)^n n a_n = 0, \tag{38}$$

which, by virtue of the matching conditions, yield

$$\sum_{n=0}^{\infty} (-1)^n a_{n+2} = \frac{\Lambda_1}{3}R^2 - \frac{4m_1}{R} \tag{39}$$

and

$$\sum_{n=0}^{\infty} (-1)^n (n+2) a_{n+2} = 2\left(\frac{\Lambda_1}{3}R^2 - \frac{m_1}{R}\right). \tag{40}$$

[8] It may be possible to demonstrate that analyticity or some degree of differentiability is unavoidable, since H_2 is just a function of one variable that is only defined in a closed region, namely $r \in [0, R]$, and along the interval it is fixed to be finite. Otherwise, this instance serves to perform a study of the possible interiors.

It is clear that there are infinitely many possible candidates for these interiors. In the following sections we will analyze in more detail the properties that the big family of candidates share in common.

A final remark is in order. The conditions at the origin to make the final solution "everywhere regular", aside from the matching being also fulfilled, are important. For instance, $H = A + Br$, A and B constants, can immediately satisfy the junction conditions, therefore the metric is C^1 in these coordinates. However G_{11} and G_{22} yield

$$G_{11} = -\left(\frac{A}{r^2} + \frac{2B}{r}\right), \quad G_{22} = -\frac{B}{r},$$

which is *not* regular at the origin, for whatever values of A or B different from zero. Of course, we could afford to study within these models a bigger family which also included singular models at the origin. In this work, as said elsewhere, we will just focus on regular models, because of the interest in studying their plausibility. The singular situation will be considered elsewhere.

7 Isotropization

To summarize, a basic property of any candidate to an interior solution is its behavior at the origin. First, due to the regularity conditions, all Riemann invariants turn out to be finite at the origin, as well as the components of the Einstein tensor, in accordance with previous works [10, 19, 18, 29]. Moreover, it is also very important to further analyze how they behave near the origin.

Taking into account the expression of H_2 in powers of $\tilde{r}$, we get

$$G_{11} = -\frac{1}{R^2}\sum_{l=2}^{\infty}(l+1)b_l\tilde{r}^{l-2}, \tag{41}$$

and

$$G_{22} = -\frac{1}{R^2}\sum_{l=2}^{\infty}\binom{l}{2}b_l\tilde{r}^{l-2}. \tag{42}$$

It is then clear that G_{11} and G_{22} are different from each other.[9] Yet we have the very relevant property that, for any of these spacetimes, it holds

$$\lim_{\tilde{r}\to 0} G_{11} = \lim_{\tilde{r}\to 0}(-G_{00}) = \lim_{\tilde{r}\to 0} G_{22} = \lim_{\tilde{r}\to 0} G_{33} = -\frac{3b_2}{R^2}. \tag{43}$$

[9] It is important to notice that the conservation of the Einstein tensor implies a relation between G_{11} and G_{22} of the type: $G_{22} = G_{11} + G'_{11}r/2$. The same holds for the stress-energy tensor.

Whence, we see that a general isotropization *independent of the model* is actually accomplished in a completely natural way. In terms of a_l we get

$$G_{11} = -\frac{1}{R^2}\sum_{M=0}^{\infty} A_M \tilde{r}^M, \tag{44}$$
$$A_M = (-1)^M (M+3) \sum_{l=M+2}^{\infty} (-1)^l \binom{l}{l-2-M} a_l,$$

and

$$G_{22} = -\frac{1}{R^2}\sum_{M=0}^{\infty} \frac{M+2}{2} A_M \tilde{r}^M. \tag{45}$$

So that

$$\lim_{\tilde{r}\to 0} G_{11} = \lim_{\tilde{r}\to 0}(-G_{00}) = \lim_{\tilde{r}\to 0} G_{22} = \lim_{\tilde{r}\to 0} G_{33} = -\frac{A_0}{R^2},$$
$$A_0 = 3\sum_{l=2}^{\infty}(-1)^l \binom{l}{l-2} a_l.$$

On the other hand, making no further assumptions on the coefficients of H_2, we can isolate two of them in terms of the rest. For simplicity, we shall isolate a_2 and a_3. The result is

$$a_2 = -\frac{10m_1}{R} + \frac{\Lambda_1}{3}R^2 + \sum_{l=4}^{\infty}(-1)^l (l-3) a_l,$$
$$a_3 = -\frac{6m_1}{R} + \sum_{l=4}^{\infty}(-1)^l (l-2) a_l.$$

With this in hand we can write the previous expression for the central value in terms of a_l, $l \geq 4$,

$$G_{11}(0) = G_{22}(0) = -\frac{3}{R^2}\left[\frac{8m_1}{R} + \frac{\Lambda_1}{3}R^2 + \sum_{l=4}^{\infty}(-1)^l \frac{(l-3)(l-2)}{2} a_l\right]. \tag{46}$$

The same can be done for the expression of H_2 in powers of $\tilde{r}$, $H_2 = \sum_{l=2}^{\infty} b_n \tilde{r}^r$. We obtain

$$G_{11}(0) = G_{22}(0) = -\frac{3}{R^2}\left[\frac{8m_1}{R} + \frac{\Lambda_1}{3}R^2 + \sum_{l=4}^{\infty}(l-3) b_l\right], \tag{47}$$

where we have used the junction conditions, i.e.

$$\sum_{l=2}^{\infty} b_n = H_1(1), \quad \sum_{l=2}^{\infty} n b_n = H_1'(1). \tag{48}$$

7.1 Examples

We will consider four examples. The first two constitute the well-known proposals of [18, 19, 44] and [20, 21]. The other two constitute a family of new candidates that naturally arise from the preceding expressions. We start with the second pair

7.1.1 Two arbitrary powers

We choose that only two specific powers of $H(r)$, say M, N, be present in this case. In order to fulfill the regularity conditions, both must satisfy M, $N \geq 2$. However, if it holds that M, $N > 2$, the Einstein tensor, G, becomes zero at the origin. Thus, if we wish a de Sitter-like behavior at, and near, the origin, we must impose one of them to be equal to 2. We then have $M = 2$, $N \geq 3$ as the only suitable choice for a two power case. $H_2(\tilde{r})$ reads

$$H_2(\tilde{r}) = b_2 r^2 + b_N r^N, \quad N \geq 3, \tag{49}$$

with

$$b_2 = \frac{2m_1}{R}\left(\frac{N+1}{N-2}\right) + \frac{\Lambda_1 R^2}{3}, \tag{50}$$

$$b_N = -\frac{6m_1}{R}\frac{1}{(N-2)}, \quad N \geq 3. \tag{51}$$

$G_{11}(\tilde{r})$ and $G_{22}(\tilde{r})$ read

$$G_{11}(\tilde{r}) = -\Lambda_1 + \frac{6m_1}{R^3}\left(\frac{N+1}{N-2}\right)(\tilde{r}^{N-2} - 1), \tag{52}$$

$$G_{22}(\tilde{r}) = -\Lambda_1 + \frac{6m_1}{R^3}\left(\frac{N+1}{N-2}\right)\left(\frac{N}{2}\tilde{r}^{N-2} - 1\right). \tag{53}$$

Whence one readily sees that their finite value at the origin coincides, as expected,

$$G_{11} = G_{22} = -\Lambda_1 - \frac{6m_1}{R^3}\left(\frac{N+1}{N-2}\right), \tag{54}$$

and it does not change significantly if the power N changes.

7.1.2 Lowest powers

This example chooses the case in which H_2 has the minimal power dependence. This amounts to taking $N = 3$ in the preceding example or setting $a_l = 0$, $l \geq 4$, in the general expressions. Its interest lies in considering the simplest situation. The result is

$$H_2(\tilde{r}) = \left(\frac{8m_1}{R} + \frac{\Lambda_1 R^2}{3}\right)\tilde{r}^2 - \frac{6m_1}{R}\tilde{r}^3, \tag{55}$$

and, near the matching hypersurface,

$$H_2(\tilde{r} - 1) = \left(\frac{2m_1}{R} + \frac{\Lambda_1 R^2}{3}\right) + \left(\frac{-2m_1}{R} + \frac{2\Lambda_1 R^2}{3}\right)(\tilde{r} - 1)$$

$$+\left(\frac{-10m_1}{R}+\frac{\Lambda_1 R^2}{3}\right)(\tilde{r}-1)^2-\frac{6m_1}{R}(\tilde{r}-1)^3. \tag{56}$$

And

$$G_{11}=-\left(\Lambda_1+\frac{24m_1}{r^3}\right)+\frac{24m_1}{R^3}\tilde{r}=-\Lambda_1+\frac{24m_1}{R^3}(\tilde{r}-1), \tag{57}$$

$$G_{22}=-\left(\Lambda_1+\frac{24m_1}{r^3}\right)+\frac{36m_1}{R^3}\tilde{r}=\frac{12m_1}{R^3}-\Lambda_1+\frac{36m_1}{R^3}(\tilde{r}-1). \tag{58}$$

G_{11} tends to $-\Lambda_1$ as $\tilde{r}$ tends to 1, in accordance with Israel's conditions.

7.1.3 The approach of Israel and Poisson

In reference [18] a plausible candidate for the energy-matter content of the interiors of regular black holes was proposed. Because of the impossibility of matching directly the de Sitter spacetime with the Schwarzschild one, the authors proposed that a singular layer of non-inflationary material should exist between the de Sitter core and the external Schwarzschild metric. The authors recognized this assumption as an attempt at overcoming the problems above mentioned. However the usual spirit of matching a stellar interior with a vacuum exterior was lost, the reason being the unavoidable presence of a singular layer, which should be placed close to the matching hypersurface and would act as a matter surface density. The authors themselves already pointed out this problem (see [44]). They considered that their approach could be improved by avoiding such a layer and by imposing a smooth transition from the hypersurface to the de Sitter core. In any case it was the only available candidate to continue the studies of quantum regular black holes. Further work, by these authors and other groups that were dealing with the same problem, addressed this idea as well as its physical consequences and properties (see e.g. [9, 19, 44]).

The task here will be to see whether this geometrical and physical model can be recovered from our analysis, *solely* based on the usual spirit of collapsing bodies, i.e. without involving *any* (singular) material layer. First of all, let us notice that, in our models, G_{00} can be re-expressed as

$$G_{00}=\frac{1}{r^2}(Hr)', \tag{59}$$

where $()'$ is the ordinary derivative with respect to r.

We now search for a solution within our family that tends to that particular solution (see, for instance, [18, 19, 10, 44]). This amounts to saying that we look for a de Sitter core for

small values of $\tilde{r}$ and a quantum contribution of the type of *the square of the characteristic curvature of Schwarzschild spacetime* near the matching hypersurface, i.e. $(G_{00})_{\rm ext.} \propto m_1^2/r^6$. This is different to assuming that in the exterior region, close to the matching hypersurface, the quantum contributions turn into a cosmological-like term. We thus have a quantum exterior different from the one encountered in the rest of the examples. Obviously, a combined model could also be solved. However we prefer to maintain the spirit of the referred works for easier comparison of our results with theirs. In fact, at the end of Sect. 8.2, we will show that the exterior vacuum contribution plays a secondary role, regardless of its precise form. All these features taken into account, we first put, generically,

$$G_{00} = \frac{1}{(B + Cr^3)^2}, \tag{60}$$

where B and C are two constants, to be determined after imposing the matching conditions as well as the mentioned physical requirements. The matching conditions lead to

$$\begin{aligned} \frac{1}{B(B + Cr^3)} &= \frac{3}{R^2} H_1(R), \\ \frac{2B - CR^3}{B(B + CR^3)^2} &= \frac{3}{R} H_1'(R), \end{aligned}$$

where H_1 comes, as usual, from the external model.

In order to reproduce the new exterior vacuum model, we see that it is possible to select it from our family, by setting $B_{\rm ext.} = 0$, and $C_{\rm ext.}^{-1} = \alpha m_1$, where $\alpha = \beta L_{\rm Pl}$, being β of order unity, and $L_{\rm Pl}$ the Planck length (α is of order unity in Planckian units). β^2 is related to the number and types of the quantized fields, [18, 44]. This choice yields

$$H_1(r) = \frac{2m_1}{r} - \frac{1}{3}\left(\frac{\alpha m_1}{r^2}\right)^2, \tag{61}$$

where we have taken into account that the exterior region is dominated by the Schwarzschild geometry, for large values of $\tilde{r}$.

Finally, B and C yield

$$B = \frac{\alpha}{6 - \dfrac{\alpha^2 m_1}{R^3}}, \quad C = \frac{2}{\alpha m_1}\left(1 - \frac{3}{6 - \dfrac{\alpha^2 m_1}{R^3}}\right). \tag{62}$$

Thus, we do obtain a model within our family that copes with the desired requirements of references [18, 19, 44]. In order to check if the solution obtained coincides in fact with that particular model, one has to solve the problem of getting R from observed or expected

values of the constants involved. This will be postponed to the following section. There, the plausibility of all these models will be analyzed in detail. To summarize, we have proven here that a –everywhere smooth— spacetime model in our family satisfies all the required geometrical assumptions as well as the particular form of G_{00} corresponding to the above mentioned references.

7.1.4 Dymnikova's model

Some time after the appearance of the previous cases a new model for a regular interior of a black hole was proposed, see [20, 21]. However the approach was now different to that of the previous authors. In order to avoid the problem of the singular layer and to introduce a smooth transition between the Schwarzschild dominated region and the de Sitter central core, the imposition of a clear matching hypersurface was relaxed. Even more, these authors did not impose any matching between the two different spacetimes at all. The problem was then that the final picture of the collapsed object departed again from the usual, expected one. Now Schwarzschild was only recovered in an asymptotical sense, for $\tilde{r}$ approaching infinity only.[10] This immediately adds a natural drawback in the model, since collapsed objects have their matter content restricted to a finite volume. However, if a sufficiently quick convergent matter model can be obtained, then the lost mass, for a observer outside the horizon of the collapsed body could become as negligible as desired. Thus one would, at least, recover a trial model, interesting enough to support or reject the conclusions of the previous authors. In this sense Dymnikova's model is quite interesting. In a later work, together with B. Soltysek, they incorporated the observational fact in favor of a non-vanishing cosmological term in the exterior region [21]. This work includes the first pure Schwarzschild exterior case also straightforwardly. We shall deal in this subsection with that model, considering an exterior Λ, or Λ_1 and a definite end of the collapsed body.

The imposition for the energy-matter content is of the form

$$G_0^0 = G_1^1 = \beta \exp\left(-\alpha \tilde{r}^3\right) + \gamma, \tag{63}$$

where the constants α, β, and γ are found after imposing our matching conditions. First of

[10] A similar assumption is made in [22] for the case of *charged*, non-rotating black holes. For the time being, here we will only consider those cases of major astrophysical interest, i.e. non-charged black holes. However, an analogous study is straightforward for their model (the core however is no longer described by the de Sitter spacetime or som well-known spacetime). A distinguished property of that model will be considered in Sect. 10.

all we can already impose that the exterior geometry is of the Schwarzschild-de Sitter type, setting $\lim_{\tilde{r}\to 1} G_{11} = -\Lambda_1$. This yields

$$\gamma = -\Lambda_1 - \beta \exp{(-\alpha)}. \tag{64}$$

We then integrate the expression of G_{11} (this is straightforward because of the assumed r^3 dependence in this model as it was also the case in the previous model) in order to obtain H_2, getting

$$H_2(\tilde{r}) = \frac{R^2}{3}\Big\{\Big(\frac{\beta}{\alpha}\Big)\Big[\frac{\exp{(-\alpha\tilde{r}^3)} - 1}{\tilde{r}}\Big] + \tilde{r}^2[\beta\exp{(-\alpha)} + \Lambda_1]\Big\}, \tag{65}$$

where we have already imposed the regularity conditions at the origin. The final step is to impose the matching conditions at the spatial hypersurface. They yield the condition

$$\beta\Big[\frac{\exp(-\alpha) - 1}{\alpha} + \exp(-\alpha)\Big] = \frac{6m_1}{R^3}. \tag{66}$$

Near the origin the expression of G_{11} tends to $\beta[1-\exp(-\alpha)]-\Lambda_1$. This constant characterizes the geometry of the core. Let us now assume that it is of de Sitter type as in the model of [21] (see also next section). Therefore, it has to be equal to an internal Λ, say Λ_2, as opposed to Λ_1, which accounts for the exterior de Sitter part. Combining both equations, one gets

$$\frac{\exp(\alpha) - \alpha - 1}{\alpha[\exp(\alpha) - 1]} = \frac{6m_1}{R^3(\Lambda_2 - \Lambda_1)}. \tag{67}$$

This equation is transcendental. In order to solve it one realizes that there always exists a value of R, big enough, in order to make $6m_1/R^3(\Lambda_2 - \Lambda_1)$ small enough. The reliability of this restriction of "big enough R" will be considered later. The numerical results of the next section will show that the associated error is indeed very small. Our aim is now to see whether the model considered by [20, 21] lies also within our family of models.

Within the restriction above, one can see that the solution is unique and appropriately given by

$$\alpha = \frac{(\Lambda_2 - \Lambda_1)R^3}{6m_1}, \quad \beta = \Lambda_1 - \Lambda_2. \tag{68}$$

The relative error being $\Delta = \alpha/[\exp(\alpha) - 1] \sim \alpha\exp(-\alpha)$. Therefore, it can be made as small as needed, provided R can be taken as large as necessary. The coefficient α can be used to define a characteristic radius, say R_c, as

$$R_c^3 \equiv \frac{R^3}{\alpha} = \frac{6m_1}{\Lambda_2 - \Lambda_1} = \frac{3R_g}{\Lambda_2 - \Lambda_1}, \tag{69}$$

where R_g is the Schwarzschild radius of the object. With this, for $0 \leq r \leq R$, G_{11} and $G_{22}(= -\Lambda_1 + \beta \exp(-\alpha\tilde{r}^3)(1 - 3\alpha\tilde{r}^3/2))$ can be rewritten as

$$G_{11} = -\Lambda_1 + (\Lambda_1 - \Lambda_2)\exp[-(r/R_c)^3], \tag{70}$$

$$G_{22} = -\Lambda_1 + (\Lambda_1 - \Lambda_2)\left[1 - \frac{3}{2}\left(\frac{r}{R_c}\right)^3\right]\exp[-(r/R_c)^3], \tag{71}$$

which exactly coincide with the expressions in the mentioned papers (see e.g. Eq. (11), (14) of [21]), although they have now been obtained for the physical range $0 \leq r \leq R$.

8 Significance and interpretation of the conditions obtained

8.1 Physical consequences of isotropization

Until now the precise value of the matching r (or R) has been overlooked. In part, because it was actually not necessary to fix it, that is, the conditions hold at any value of r, which is actually a rather remarkable fact. Despite the interesting observation that all the previous properties are independent of the precise value of R, the final aim is to match the preceding results with physical observations and with well-known theoretical behaviors of the quantum state near the origin, as well.

Isotropization is a general property, as we have pointed out, and it can be interpreted as a geometrical one. By this we mean that it is actually the Einstein tensor that becomes isotropic, regardless of the spacetime chosen, among our general family. If we write the Einstein tensor in terms of the energy-momentum tensor and the Λ contribution, we realize that, at the origin, the following relation must hold

$$G(0) = \frac{8\pi G_N}{c^2}T(0) - \Lambda_2 g(0). \tag{72}$$

That is, both the vacuum contribution and the rest of the contributions of the excited states of the quantum field add directly. However, we can break this indeterminacy with the help of some physical model of quantum behavior near the origin. According to current ideas, we expect that the contribution from the vacuum will dominate those of the rest of states. Thus we simply impose $T(0)$ to be negligible or zero when compared with $\rho_{\Lambda_2}(= (c^2/8\pi G_N)\Lambda_2)$. On the other side, $G(0)$ is in all cases a value that depends on R, m_1, and Λ_1 only. Therefore, we are faced up with a general relation of the type

$$G(m_1, \Lambda_1; R) = -\Lambda_2 g \tag{73}$$

or

$$G_{11}(m_1, \Lambda_1; R) = -\Lambda_2. \tag{74}$$

This relation will allow us to find R in terms of m_1 and $\Lambda_2 - \Lambda_1$ only. The expressions of G_{11} in more general situations have been given before.

Finally, let us add that the model chosen here can be reinterpreted as having imposed an isotropization of the quantum gravitational effects in favor of the quantum vacuum. It is a new isotropization added to the geometrical one, which in our family has a universal character.

8.2 Numerical results

According to several recent observations [38, 39], in what follows we shall assume that $\rho_{\Lambda_1} \in [10^{-10}, 2\times 10^{-8}]$ erg$\cdot$cm^{-3} . However an analysis shows that essentially the same results will hold even if ρ_{Λ_1} is *zero*, or another type of quantum vacuum contribution is assumed, as was the case in the Israel-Poisson model. The fundamental contribution comes directly from the quantum gravitational model that is imposed for the core. For the reasons given before, we shall consider, for the time being, an internal de Sitter-like core. Nonetheless, there is no present agreement about the scale at which regularization could act. This is directly connected with the so called "*Cosmological Problem*", which is the subject of intense research currently.[11] Our strategy will be to let this constant free and then consider the different results that will come out. A convenient way to handle and integrate this indeterminacy is to set $\Lambda_2 = 10^{3s}\Lambda_1$, being $3s$ the free parameter that governs the scale of renormalization. For instance, if s is around 40, we are then considering that regularization takes place at Planck scales, etc.

8.2.1 Two arbitrary powers

The fundamental relation is

$$\Lambda_1 = \Lambda_2 + \frac{6m_1}{R^3}\left(\frac{N+1}{N-2}\right), \quad \forall N \geq 3, \tag{75}$$

[11]The possibility of dealing with two Λ's has not been stressed yet and will not be discussed here. Elsewhere, we will try to perform a detailed theoretical analysis on the plausibility of having $\Lambda_1 - \Lambda_2$ of the order of the used numerical (Λ_1) and observational (Λ_2) values.

m_1	$s=30$	$s=40$	$s=50$
$M_\odot$	10^{-9}	10^{-19}	10^{-29}
$10^3\,M_\odot$	10^{-8}	10^{-18}	10^{-28}
$10^6\,M_\odot$	10^{-7}	10^{-17}	10^{-27}
$10^9\,M_\odot$	10^{-6}	10^{-16}	10^{-26}

Table 1: R in cm for various astrophysical and galactic objects and different scales of regularization ($s=30$ corresponds to a GUT's regularization scale, $s=40$ to a Planckian one, etc.). In any case $R/L_{\rm Reg}$ is much bigger than 1 ($R/L_{\rm Reg}\sim 10^{(-6+s/2)}$). Therefore all of them are mainly in the semiclassical regime.

whence

$$R = R_\odot \sqrt[3]{M}\sqrt[3]{\frac{N+1}{4(N-2)}}, \tag{76}$$

where we have put $m_1 = M m_\odot$, $m_\odot$ being the Sun mass, and $R_\odot \equiv \sqrt[3]{24 m_\odot/(\Lambda_2-\Lambda_1)}$. This value only depends on the regularization scale and, in fact, is in correspondence with the solution for a collapsed object of one solar mass in the case of the "lowest powers" model: $R_\odot \in [3\times 10^{21-s}, 6\times 10^{20-s}]$cm. For $s=40$ we get $R_\odot \in [3\times 10^{-19}, 2\times 10^{-20}]$cm. Yet we see that the object has a quantum size very far from Planckian scales even if s is bigger. In general $R_\odot/L_{\rm Pl} \geq 10^{13}$! Moreover this result is valid for all N since for any N we have that $R\in[0.6,1]R_\odot\sqrt[3]{M}$. It is obvious that for any astrophysical object the final properties are very similar. Table 1 comprises different massive objects and regularization scales.

8.2.2 Israel-Poisson's approach.

We have found that the corresponding model within our family must satisfy

$$B = \frac{\alpha}{6-\dfrac{\alpha^2 m_1}{R^3}}. \tag{77}$$

In this case, $B^{-2} = \lim_{\tilde r\to 0} G_{00} = \Lambda_2$, so that

$$R^3 = \frac{\alpha^2 m_1}{6-\alpha\sqrt{\Lambda_2}} = \frac{\beta^2}{6-\beta\sqrt{\Lambda_2 L_{Pl}^2}} m_1 L_{\rm Pl}^2. \tag{78}$$

This model clearly depends on the coefficient β. For instance, in order to obtain a solution, we must have $\beta^2 < 36/(\Lambda_2 L_{\rm Pl}^2)$. The natural scale of regularization in this model is the

Planckian one since from the beginning the coefficient α was related to the Planck length. Obviously other regularization scales would simply change $L_{\rm Pl}$ by the corresponding scale. Using standard values for Λ_2 that use a Planckian regularization scale, and that β^2 should be at most of order unity [18, 44], we get $R \sim \sqrt[3]{M} \times 10^{-20}$cm. This result is in complete agreement with the foregoing values, even though the models are very different, from a physical point of view.

8.2.3 Dymnikova's model

In the original model there is no frontier (hypersurface) up to which the spacetime is of exact Schwarzschild-de Sitter type. Thus a radius cannot be defined at which the collapsed object ends. However, a characteristic radius, R_c, is found there, and it is proven that, for $(r/R_c)^3 >> 1$, the outside matter can be dismissed in comparison with the inner one. This was the main reason in order to consider that such model describes approximately an object of quantum size.

In building an analogue of that situation, we decided to introduce three parameters, namely α, β, and γ in order to better mimic the model for G_{11}. In the rest of the models there are only two parameters. Now we have an extra freedom. In fact, for any $R > \sqrt[3]{2}R_c$, there exists a unique solution for them, provided Λ_1 is less than Λ_2, as we expect. For the rest of the values, there is no solution at all. Thus we are faced with an expected freedom in R that can be dismissed adding a further assumption. For the time being let us see that there are solutions which fit in a very precise way with the previous results.

This model was solved explicitly for the case in which α was big enough so that the relative error, $\alpha \exp(-\alpha)$, was small. Let us redefine R as $R = k\sqrt[3]{M}R_\odot$, where k is an adimensional constant, to be found below. Thus

$$\alpha = \left(\frac{R}{R_c}\right)^3 = 4k^3. \tag{79}$$

The relative error is then $\Delta = 4k^3 \exp(-4k^3)$. Whence, we see that choosing k of order unity the relative error is very small (for instance, for $k = 2$ it is only of 10^{-11} %!). Thus $R = k\sqrt[3]{M}R_\odot$, with k of order one, are all very good solutions of our corresponding model. Since there is no well defined frontier for the collapsed object in the approach of [20, 21], a well-defined R does not exist. There, the assumption about the characteristic size of the object was ascribed to R_c. In fact, $R_c = R/(k\sqrt[3]{4})$, so that the results in [20, 21] fit well with ours.

Finally, we may add a further restriction in our model in order to eliminate the freedom in R. Before we proceed, it is worth remarking that the above discussion was just a check of the issue, if a model of the type [20, 21] was inside our family. In fact, this is only a completion criteria that allows us to continue dealing with this model. It is commonly thought that no known quantum fields can be responsible for such a G_{11}.

A natural way to put an end to the indeterminacy in R is by setting, e.g., $\gamma = 0$. With this assumption we have chosen an option that completely separates the quantum behavior in both regions. Obviously, many other choices are possible. The result of such choice is

$$R^3 = \frac{6m_1 \ln(\Lambda_2/\Lambda_1)}{\Lambda_2 - \Lambda_1 - \Lambda_1 \ln(\Lambda_1/\Lambda_2)}. \tag{80}$$

Now, taking into account that $\Lambda_2 = 10^{3s}\Lambda_2$, and recalling that we expect $s \geq 40$, we obtain

$$R \sim 2R_c \sqrt[3]{s}. \tag{81}$$

For $s = 40$ this yields $R \sim 7R_c$, which again falls within the same order as all the previous models. This fact leads us to believe that the main properties found hitherto are rather general and independent of the model that describes the collapsed object. Notice finally that the initial considerations that came from a direct, and wrong, matching between a de Sitter core and a Schwarzschild exterior yielded $R = R_c$. So we can assert that the correct solution modifies the naive, and wrong, expected value by a small factor. This is quite an amazing conclusion.

9 Horizons and an interpretation of the regularized black hole

Looking, for instance, at Eq. (30) in [21] and comparing it with our result

$$g_{00} = -1 + H_2, \tag{82}$$

we realize that substituting here our corresponding H_2 for their model, these expressions turn out to be very similar same, except for an overall sign due to the different signatures ((+,–,–,–) instead of our (–,+,+,+)). We conclude that the same structure for the horizons and Cauchy hypersurfaces is obtained. In [21] the solutions are obtained by approximations of the exact solution, so that these results and ours are really coincident (the relative error being completely negligible with respect to both exact solutions).

In general, the horizons result from the cancellation of g_{00}. Thus we are left with a general set of horizons. A global study, for all the candidates encountered, has not been carried out yet. We could focus on examples, and try then to extract some general features from them, but we do not find this of primary importance.[12] The main point is here, in fact, that the matching occurs at a radius which is substantially smaller than the Schwarzschild radius of the object. Therefore we will always have a typical exterior, a vacuum transition region extending until the matching with the object happens, and a quantum-dominated interior, which finally converges to a de Sitter core. In the vacuum interior region and in some part of the quantum object, the role of t and r are not interpreted as usual (∂_t changes its character). This is the reason for adequately treating the horizons: to see where exactly such changes appear. But, we can still perfectly agree in ordinary physical terms without requiring a general resolution of the precise radii at which horizons occur.

Another issue is that of the topology of the solutions, if they are regular black-holes, and their possible extensions, i.e. "universe reborn". Its general structure can be found in [45], where it was shown that the topology of any regular black hole should be similar to that of a singular-free Reissner-Nordström spacetime. Thus, there appears a necessary topology change, if one deals with their complete extension.

10 Energy conditions

A common point in dealing with the avoidance of singularities is to show that the required energy conditions in the singularity theorems (see e.g. [6]) fail to be valid.

Here we will study three local energy conditions, commonly considered in the literature: strong energy conditions (SEC), weak energy conditions (WEC) and null energy conditions (NEC). SEC are related with the formation of singularities in the collapse of an object. WEC are directly related with the energy density measured by an observer and NEC are useful in order to include some spacetimes which violate the first two, but are predicted by some quantum models, e.g. anti de Sitter spacetime. Although an analysis of energy conditions helps to understand the physics of a model, one has to be cautious ascribing to them more relevance than they actually have. In several systems, mainly when quantum effects play a fundamental role, they all may be violated with less difficulty (see e.g. the review in [46]).

[12] With respect to the other models found here, we have got that the results are rather similar to those of Dymnikova and Soltysek's model [21].

Our results also agree with those in [29], where the authors study them in general spherically symmetric spacetimes.

Let $\{\vec{e}_a\}$, $a = 0, 1, 2, 3$, be a dual vector basis of the cobasis in (5), defined by $\mathbf{\Theta}^b \vec{e}_a = \delta^b_a$, $b = 0, 1, 2, 3$. Any timelike vector field, $\vec{V}$, in this manifold can be represented by

$$\vec{V} = A^b \vec{e}_b, \qquad (A^0)^2 = 1 + \sum_{i=1}^{3} (A^i)^2, \tag{83}$$

where A^b are some functions.

On the other hand, from the results of Sect. 2, the Ricci tensor is

$$\mathbf{Ricci} = R_{00}(\mathbf{\Theta}^0 \otimes \mathbf{\Theta}^0 - \mathbf{\Theta}^1 \otimes \mathbf{\Theta}^1) + R_{22}(\mathbf{\Theta}^2 \otimes \mathbf{\Theta}^2 + \mathbf{\Theta}^3 \otimes \mathbf{\Theta}^3), \tag{84}$$

where $\otimes$ is the tensor product. A similar expression is valid for the Einstein tensor.

SEC require $R_{VV} \equiv R_{ab} V^a V^b \geq 0$, for all $\vec{V}$. From the expressions above, we obtain $R_{VV} = R_{00} + (R_{00} + R_{22})[(A^2)^2 + (A^3)^2]$. Taking into account Eqs. (7) and (9), $R_{VV} = G_{22} + (G_{00} + G_{22})[(A^2)^2 + (A^3)^2]$. Finally, using Einstein's equations, (12), and the fact that A^2, A^3 are free, we get

$$\text{SEC} \leftrightarrow \rho + p_2 \geq 0, \quad p_2 \geq 0 \tag{85}$$

where ρ is the energy density measured by $\vec{e}_0$, $8\pi\rho = G_{00}$ and p_2 is the tangential pressure (or stress) of the source, $8\pi p_2 = G_{22}$. This is the usual representation of SEC. However, the GNRSS family allows for a different, more useful, expression. Indeed, as mentioned elsewhere, it is easy to show that, for a GNRSS spacetime, $p_2 = -(\rho + r\rho'/2)$, where $()' \equiv d()/dr$. Therefore, we can eventually write

$$\text{SEC} \leftrightarrow p_2 \geq 0, \quad \rho' \leq 0. \tag{86}$$

Following analogous steps, one finds, for WEC ($G_{VV} \geq 0$, for all $\vec{V}$)

$$\text{WEC} \leftrightarrow \rho \geq 0, \quad \rho' \leq 0. \tag{87}$$

In the case of NEC, $\vec{V}$ is a *null* vector field, $\vec{V} \cdot \vec{V} = 0$, and requires the evaluation of $R_{ab} V^a V^b = G_{ab} V^a V^b \geq 0, \forall \vec{V}$. One obtains

$$\text{NEC} \leftrightarrow \rho' \leq 0. \tag{88}$$

Thus one sees, that a necessary condition *common to all of them* is that the energy profile of the sources be a non-increasing function. Moreover, in WEC, $\rho \geq 0$ conveys the positivity

of energy density whereas, in SEC, $p_2 \geq 0$ refers to the positivity of tangential pressures. The last is in fact violated from some value of r downwards, as can be readily seen from $p_2 = -(\rho + r\rho'/2)$ and the fact that the source is regular at the origin. Therefore, eventually, the singularity is not created.

10.1 Energy conditions for different models

It is now easy to evaluate the fulfillment or violation of the energy conditions in the models presented before. It turns out that WEC, and NEC, are satisfied in all of them very easily for any value of r (e.g. for (anti) de Sitter core, $\rho' = 0$). One only needs to impose $\Lambda_1 > \Lambda_2$. On the other hand, as mentioned before, SEC are violated in all of them. The new result is that this occurs far away from the regularization scale. Indeed, a detailed analysis shows that SEC are violated for $r \leq R_{SEC}$, with

$$R_{SEC} = \sqrt[N-2]{\frac{2}{N}}R, \quad R_{SEC} = \left(\sqrt[3]{\frac{\alpha^2 m_1}{4(6R^3 - \alpha^2 m_1)}}\right)R, \quad R_{SEC} = \sqrt[3]{\frac{2}{3}}R_c, \tag{89}$$

where, all the quantities have been defined in Sect. 7.1 and the solutions correspond to the two-power model, the Israel-Poison's model and Dymnikova's model, respectively. It is clear from these results, that SEC are violated in the most part of the object, i.e. $R_{SEC} \lesssim R$ (see also [20]). For the evaluation of the Israel-Poison's model, we have used the same numerical values as in Sect. 8.2.

Let us now consider the series of models in [22]–[24]. In [22] the first regular models for non-rotating black holes were given. We remark that all these solutions are *charged* black holes, e.g. $|e|/m$ is of order unity, where e is the charge of the black hole. This is far away from astrophysical observations of black holes (see e.g. [40]). Nevertheless, these solutions are very important in this issue, since they are the *first* solutions to regular black holes with a clear interpretation in terms of a known quantum field, in this case nonlinear electrodynamics. A direct analysis of those works shows that, unexpectedly, *all* of them belong to the GNRSS family of spacetimes. Let us choose one of the three solutions proposed.

For instance, the one in [24] sets, in our notation,

$$H = \frac{2m}{r}\left[1 - \tanh\left(\frac{e^2}{2mr}\right)\right].$$

The energy density can be computed from Eqs. (9). We obtain

$$8\pi\rho_{AG} = \frac{e^2}{r^4\cosh^2(e^2/2mr)} = \frac{1}{e^2 s^4}\frac{1}{y^4\cosh^2 y^{-1}},$$

where, in the last expression we have used the same notation as the authors, that is, $s \equiv |e|/2m$ and $y \equiv 2mr/e^2$. It is easy to see that ρ_{AG} is always positive. However, ρ'_{AG} becomes positive from some y_0 downwards. In fact, a direct computation shows that the maximum energy density is achieved for $y_0^{-1} \tanh(y_0^{-1}) = 2$, i.e. $y \sim .4842$, and that ρ is an increasing function for $y \leq y_0$. Finally, taking into account [24] that $|e|$ must be lower than $1.05m$, in order to have a regularized solution, we can readily show that for any $r \leq 0.2664m$, that is, the *most* part of the object, *all* energy conditions fail, even WEC and NEC. The rest of proposed models behave in an identical way.[13] This adds a new (elementary) example to the violation of energy conditions models when quantum effects play an important role (see [46] for a recent review) and shows us that energy conditions help understanding the models, but not necessarily should bound the search for new solutions.

11 Quantum aspects of the models

To find the sources for the models presented is not easy, but this will be a necessary step to undertake in order to show the plausibility for the avoidance of the singularities. To reach this goal within our scheme, one has in fact to quantize the sources. The symmetries of the matter-energy tensor, i.e. $\rho + p = 0$, where p is the radial pressure (or stress) —$8\pi p = G_{11}$— and the one coming from spherical symmetry, $p_2 = p_3$ are fundamental in order to probe which type of quantum fields could correspond to their sources. Taking into account the spherical symmetry of the spacetime and, consequently, of the Einstein tensor, one easily checks that all classical fields adapted to the spherical symmetry satisfy this condition (see for instance Sect. 3.8 of [47]). This bonus was indeed expected, because spherical symmetry is a basic geometric symmetry in the scheme. On the other hand, the other Eq., $\rho + p = 0$, can be rewritten in a general covariant way, without referring to any special set of observers. Its expression is then

$$T_{\ell\ell} \equiv T_{\mu\nu} l^\mu l^\nu = 0, \tag{90}$$

[13] In [22] the authors even give a graphical representation of $-g_{tt}$, whence direct visualization proves our assertions.

where $\vec{\ell}$ is the geodesic radial null direction characteristic of the GRNSS spaces. If one considers a scalar field, one has

$$\begin{aligned} T_{\alpha\beta} &= (1-2\xi)(\nabla_\alpha\phi)(\nabla_\beta\phi) + (2\xi - \tfrac{1}{2})g_{\alpha\beta}(\nabla^\rho\phi)(\nabla_\rho\phi) \\ & -2\xi\phi\nabla_\alpha\nabla_\beta\phi + \tfrac{2}{n}\xi g_{\alpha\beta}\phi\,\Box\,\phi - \xi\Big[G_{\alpha\beta} + \tfrac{2(n-1)}{n}\xi R g_{\alpha\beta}\Big] \\ & +2\Big[\tfrac{1}{4} - (1-\tfrac{1}{n})\xi\Big]m^2 g_{\alpha\beta}\phi^2, \end{aligned} \tag{91}$$

where ξ is a constant representing the coupling between the scalar and the gravitational field, ϕ is the (classical) scalar field, and n is the dimension of spacetime. Whence we obtain

$$T_{ll} = (1-2\xi)\dot{\phi}^2 - 2\xi\phi\ddot{\phi}, \tag{92}$$

where $()^{\cdot} \equiv \ell^\lambda\nabla_\lambda$, and where we have used the fact that $\boldsymbol{\ell}$ is null and geodesic. If the field were a classical one, we should have

$$(1-2\xi)\dot{\phi}^2 - 2\xi\phi\ddot{\phi} = 0. \tag{93}$$

The first thing one notices is that there is no ξ that makes Eq. (93) be identically satisfied. For instance, in the conformal case, the trace of T is the central object. Then, choosing $\xi = -(n-2)/4(n-1)$, the conformal coupling, and $m = 0$, one gets that the trace vanishes for the classical field. Therefore, Eq. (93) imposes now more conditions on ϕ than in the well-known conformal case. This is a remarkable difference with respect to conformal field theory. Actually Eq. (93) can be integrated, giving two results depending on whether $\xi = 1/4$ or $\xi \neq 1/4$. In any case, they must satisfy the field equations.

For the GNRSS family, the free scalar field equations, $(\Box + m^2 + \xi R)\phi = 0$, where $\Box$ is the d'Alembertian operator of each spacetime, may be written as ($l = 0, 1, \ldots$)

$$\begin{aligned} -(1+H)\partial_t^2\phi + 2H\partial_t\partial_r\phi + (1-H)\partial_r^2\phi - \partial_r H(\partial_t\phi - \partial_r\phi) \\ + \Big[m^2 + \xi R - \tfrac{l(l+1)}{r^2}\Big]\phi = 0, \end{aligned} \tag{94}$$

where R, the scalar curvature, is given by

$$R = \frac{2H}{r^2} + \frac{4\dot{H}}{r} + \ddot{H}, \tag{95}$$

and where a separation of variables has been done using the spherical symmetry of the problem. Hence, ϕ in Eq. (94) is a function of t and r only. We notice that R and Eq. (94) may be written more intrinsically as $R = (2/\theta^2)\ddot{(H\theta^2)}$, where $\theta = 1/2r^2$ is the expansion of $\boldsymbol{\ell}$, and

$$2\partial_u\dot{\phi} - 2\dot{H}\dot{\phi} - 2H\ddot{\phi} + \Big[m^2 + \xi R - \frac{l(l+1)}{r^2}\Big]\phi = 0, \tag{96}$$

where $u \equiv (1/\sqrt{2})(r - t)$. It turns out eventually that the solutions of Eq. (93) do not satisfy the free scalar field equations. Furthermore, the end result is that no free classical field satisfies Eq. (93).

In order to see whether a quantum treatment could make Eq. (93) be satisfied, we shall begin the process of quantization of the sources for the GNRSS family (we remark that actually the steps are valid for *any* Kerr-Schild-like spacetime). First, one needs to calculate δ. In fact, each spacetime of the GNRSS family is obtained by changing $H(r)$, and leaving $\boldsymbol{\ell}$ fixed. This is actually how the different interiors are obtained. Obviously, since $\boldsymbol{\ell}$ is null, we could also choose that $\boldsymbol{\ell}$ changes to a multiple of it, but this is unnecessary choosing a convenient definition of H. Therefore, $\delta g_{\mu\nu}$ is simply

$$\delta g_{\mu\nu} = 2(\delta H)\ell_\mu \ell_\nu, \tag{97}$$

and, because $g^{\alpha\beta} = \eta^{\alpha\beta} - 2H\ell^\alpha\ell^\beta$, we also have $\delta g^{\alpha\beta} = -2(\delta H)\ell^\alpha\ell^\beta$. Finally, $g = \det(g_{\alpha\beta}) = \det(\eta_{\alpha\beta} + 2H\ell_\alpha\ell_\beta) = \det(\eta_{\alpha\beta}) = -1$.

The following step is to write down the action, $S[g_{\alpha\beta}]$, in terms of $S[\eta_{\alpha\beta}]$. By definition, see e.g. §6.3 in [47],

$$T_{\alpha\beta}(x) = \frac{2}{[-g(x)]^{\frac{1}{2}}} \frac{\delta S}{\delta g^{\alpha\beta}(x)}, \tag{98}$$

where S corresponds to the classical action of matter. Performing a standard calculation, we get

$$S[g_{\alpha\beta}] = S[\eta_{\alpha\beta}] - \int T_{\ell\ell}(g_{\alpha\beta})\delta H \, d^n x. \tag{99}$$

Whence,

$$T_{\ell\ell}[g(x)] = -\frac{\delta S[g]}{\delta H}\Big|_{H=0}. \tag{100}$$

This result may be straightforwardly generalized to include any spacetime which is in a Generalized Kerr-Schild (GKS) correspondence with another one, i.e. $\bar{g}_{\alpha\beta} = g_{\alpha\beta} + 2H\ell_\alpha\ell_\beta$ (see e.g. [30]). Notice that the GKS form clearly takes the aspect of a problem with a spacetime acting as a background, not necessarily flat spacetime.

In our case, we have $T_{\ell\ell} = 0$ (in fact, this is the case of, e.g., any Kerr-Schild metric with a geodesic $\boldsymbol{\ell}$. Therefore, the following results are also valid, for instance, for Kerr-Newman metrics, or some deviations of them, which play a similar role as the GNRSS here and have been recently proposed, [28]). Thus, should any classical field exist, this would mean that the

classical action should remain invariant under a Kerr-Schild relation. We have already shown that this possibility is forbidden by Eqs. (93) and (94) or (96). Nevertheless, expression (100) tells us that $T_{\ell\ell}$ becomes now the essential quantity in the process of quantization, in the same footing as T^{λ}_{λ} in the scalar case. This result points towards the structure of a quantization of a GKS spacetime. Its study is, as of now, in progress and will be a matter of further research. It is clear that a complete solution of the problem in hand depends on this issue.

The aim is now to test if the interiors can be the result of the quantization of some classical field. To that end, it is enough to focus the attention on the scalar field. We will therefore study the possibility that $< T_{\ell\ell} >= 0$. This is a natural option, due to the non-existence of a triad (ϕ, ξ, m) satisfying Eq. (93) and Eq. (96) at the same time. Of course, this is not always a solution. Namely, given a spacetime metric, $g_{\alpha\beta}$, its Einstein's tensor, $G_{\alpha\beta}$, will not very often correspond to a known physical source. It is a general fact of any classical field equations (e.g. the monopole solution of classical electromagnetism). The same may happen now, even after having quantized the field. Then, there still would be a place for a new theory that explains it, or, eventually, it should be considered as a non-physical solution. It is interesting that some of the GNRSS solutions are solutions of a non-linear generalization of electrodynamics, see [22], which is also connected with string/M-theory.

Coming back to the point, we have that $< T_{\ell\ell} >$ should vanish, while $(T_{\ell\ell})_{\rm Cl}$, the classical contribution, is different from zero for any spacetime belonging to the GNRSS family.

Let us compare our present situation with that of conformal field theory and the de Sitter spacetime. There we have that for certain values of ξ and m, say the conformal coupling and $m = 0$, the classical field verifies

$$(T^{\lambda}_{\lambda})_{\rm Cl} = 0. \tag{101}$$

However, one finds after a standard calculation ($n = 4$, $\hbar = 1$), [47, 48],

$$< T^{\lambda}_{\lambda} >= -\frac{a_2}{16\pi^2}, \tag{102}$$

where $a_2 = -1/15\alpha^4$, and α is the "radius" of the de Sitter spacetime. In that way, the de Sitter geometry, i.e. $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$, may be interpreted, from a source point of view, as the semiclassical effect of the quantum vacuum associated with the conformally coupled, massless, scalar field. In fact, there are several other possibilities, for other couplings and masses, but the effect of the anomaly is most clearly seen in the conformally coupled and massless case. However, $T_{\ell\ell}$ obeys the contrary pattern, i.e. $(T_{\ell\ell})_{\rm Cl} \neq 0$, while we pretend

that $< T_{\ell\ell} >= 0$. We shall show below that this behavior actually comes from an *identical* anomaly effect.

For the de Sitter spacetime, one has

$$< T_{\alpha\beta} >= \frac{T}{4} g_{\alpha\beta}, \tag{103}$$

where $T \equiv < T^\lambda_\lambda >$. Therefore, since de Sitter spacetime belongs to the GNRSS family, $(T_{\ell\ell})_{\rm Cl}$ is different from zero, yet we have $< T_{\ell\ell} >= 0$. And this result has been obtained using the *same* standard formalism as that of the semiclassical approach to gravity. The reason is to be found in the fact that, for any local observer, we can write

$$< T_{\alpha\beta} >= \begin{pmatrix} \Lambda & & & \\ & -\Lambda & & \\ & & -\Lambda & \\ & & & -\Lambda \end{pmatrix}, \tag{104}$$

and, therefore, $< T^\lambda_\lambda >= -4\Lambda \neq 0$, while $< T_{\ell\ell} >= T_{00} + T_{11} = 0$. Thus, the same reasoning that leads from $(T^\lambda_\lambda)_{\rm Cl}$ to $< T^\lambda_\lambda >$ explains, for the de Sitter spacetime, that $(T_{\ell\ell})_{\rm Cl}$ is non zero, but $< T^\lambda_\lambda >$ vanishes.

Finally, we remark that all these models tend to form a de Sitter core as $r/R << 1$. Thus, the de Sitter spacetime will be a good approximation for all them, lying still in the semiclassical regime (see table 1). In fact, in the models of Sect. 8.2, the deviation is proportional to $(r/R)^3$, and the approximation is good enough, provided that $r/R << 1$. On the other hand, the semiclassical approach makes only sense as an approximation. Therefore, we do not need to include higher order corrections, as of now.

Another point of view is that $T_{\alpha\beta}$ may be written as $T_{\alpha\beta} = (T_{\text{de Sitter}})_{\alpha\beta} + \Delta T_{\alpha\beta}$. It turns out that both $T_{\text{de Sitter}}$ and ΔT satisfy the four Wald axioms (see [49]), for a $T_{\alpha\beta}$ to be regularized. Thus, we could interpret that here we are regularizing the de Sitter's part, as if it were a background, (i.e. the vacuum contribution above), to which one should further add the contributions of the "excited" states, represented by $\Delta T_{\alpha\beta}$. All this shows, at the very least, that the general scheme of quantization fits quite well within the GNRSS family.

We recall that those above are just steps towards the confirmation of a plausible quantum origin for all (or some) of the solutions in the GNRSS family. To completely fix the issue, one should either solve Eq. (96), by finding a particular set of modes, or using Eq. (100). Regarding the resolution of Eq. (96), we note that a further separation of variables between t and r is not possible, since the solution should be valid for the region $r < R$, where the

spacetime is non-stationary, so that usual mode decomposition is useless. Another set must be found, compatible with Eq. (93), that should be matched with some of the well-known exterior ones, coming from the solution of Schwarzschild or Reissner-Nordström spacetimes. Complete fixing of the exact problem seems to be a difficult task.

Let us finally resort to a different way, less fundamental but seemingly more effective, in order to incorporate possible quantum corrections into the models. This may serve as a temporary solution, yet one that has been thoroughly used in the literature to deal with the effects of quantum fields in gravity. The interesting fact in our context is that, in this way, they are easily incorporated into *any* classical solution of the GNRSS family.

Indeed, it is only necessary to compute $< T_{00} >$ to obtain $< T_{\alpha\beta} >$. This is because any GNRSS spacetime satisfies $< T_0^0 >=< T_1^1 >$, $< T_2^2 >=< T_3^3 >$. And, additionally, conservation of $< T_{\alpha\beta} >$ yields $< T_2^2 >=< T_0^0 > +(r/2) < T_0^0 >$ (see also footnote 9). Thus, the stress-energy regularization is similar as for trivially scalar fields, where the trace of $< T_{\alpha\beta} >$ is the only piece to be regularized necessarily. This greatly simplifies the computation. Moreover, another very interesting (and important) property of the GRNSS spaces is that the solution of $T = T_1 + T_2$ is given by $H = H_1 + H_2$, where H_1, and H_2 are particular solutions for T_1, and T_2, respectively. This last property can be interpreted as a manifestation of the equivalence principle, since this is the same as asserting that $m = m_1 + m_2$, where m_i are the mass function in each case. Therefore, in order to find the (well-established) quantum corrections, it is only necessary to integrate $G[H_{\text{q.}}] =$ quantum corrections and write $H_{reg} = H_{class} + H_{q.}$ in order to know $< T_{\alpha\beta} >$.

In fact, if one just considers (as is commonly assumed, see e.g. [47, 18, 48]), that semi-classical corrections are, roughly speaking, of the form $\propto \mathcal{R}^2$, with $\mathcal{R}^2$ some geometrical Riemannian invariant of the *classical* spacetime, then the corrections are obtained through direct integration of

$$\frac{1}{r^2}(H_q r)_{,r} = \lambda \mathcal{R}^2,$$

where λ is of order $G\hbar$, i.e. of order unity in Planckian units, and is related with the number and types of quantized fields, [18, 44]. An application of this method to two well known cases, namely de Sitter and Schwarzschild, will speak for itself. In the de Sitter case we have that $R = 4\Lambda =$ const, therefore R^2 —as well as any other Riemannian invariant— is a constant and the quantum correction of a macroscopic de Sitter metric turns out to be a de Sitter term. Thus one arrives at $H_{\text{ren}} = \Lambda_{\text{ren}} r^2/3$ (where the subscript "ren" means renormalized). This is a well known result for the renormalization of a de Sitter spacetime (see e.g. [47, 48]).

The Schwarzschild case has $R = 0$, for it is a classical vacuum solution. However, $R_{\alpha\beta\lambda\mu}R^{\alpha\beta\lambda\mu}$ is $48m^2/r^6$. Thus, we set a quantum vacuum contribution of the type $\lambda m^2/r^6$. Integration yields the remarkable result that the quantum vacuum corrections due to its polarization yield a spacetime metric belonging to the GRNSS spaces with $H = 2m/r - \lambda m^2/r^4$. Other sources are to be found within the effective actions of M-theory or string theory (see e.g. [22] and references therein[14]) and could be easily incorporated into our scheme, once the corresponding Einsteinian metric is computed. We will consider them elsewhere.

12 Final remarks

The first thing to be noticed is the intrinsic freedom present in our model, which is as large as the measure of the set of analytic functions of one variable. This is a very rewarding feature, since it allows to impose further restrictions coming from new proposals. In particular, it surely will be a helpful tool when trying to find explicitly a quantum field responsible for the G_{11} and G_{22} in the funadamental uncharged case. For comparison, in all previous works, based on a unique model, the prospective of finding a quantum field related with their energy-matter content was hopeless.

The second thing to be mentioned is, that we have here been able to develop in depth some relevant issues left incomplete in other works. For instance, in [19], the authors studied the interpretation of the de Sitter region as a new universe birth. They also considered the effects of the evaporation of the black hole which would be interesting to be computed for the models presented here. In [44], the authors studied the stability of the model. In the present paper, all these statements have been substantially implemented, and this in a very natural way (see the preceding sections).

On the other hand, the source origin for both examples in the section before is clear, but they do not provide the right solution of the problem we have in hand. The de Sitter one is not enough, since a de Sitter interior cannot be directly joined with an exterior coming from a black hole. And the Schwarzschild solution is singular at the origin. Then, one has to find a deviation of the de Sitter spacetime (a non-isotropic metric) still physically meaningful, although this metric will be only needed macroscopically. This will be the subject of subsequent work (Refs. [50, 51, 52] contain hints towards this direction).

We have here given just the first steps towards the goal of the (semi-classical) quantization

[14]Indeed, it is not difficult to see that all those solutions belong to the GNRSS class.

of any suitable source. Our preliminary results show undoubtedly that this process, for the case of Kerr-Schild type of metrics, is under the same footing as when one has to deal with a conformal field theory.

We have also shown that the solutions behave as constituting an anomaly effect of their sources. Moreover, we have seen how quantum field corrections can be easily incorporated into any initial classical model within the family. A result which tells us that the quantization scheme may be carried out more easily here than in conformal field theory. All these findings not only compel us to believe that black hole singularities are likely to be removed by quantum effects but, in our view, open also a new window for the search of a compatible quantum field that, once regularized, may yield the same result for, at least, a particular T inside these models.[15]

Finally, the preceding sections were centered on the free field equations. The possibility of considering some potential is certainly important, if it can be related to some physical theory, as in [28] for the case of rotating black holes. We reckon that the field approach above may yield a different and useful strategy towards the same goal.

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[15] The rotating case, which is of major astrophysical interest, and the —highly– rotating and charged one, which may be associated with spinning particles, seem to yield results very similar to the ones presented here, see e.g. [28]. This is also the outcome of preliminary calculations of ours, to be reported elsewhere [53].

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